ON σ-FINITE INVARIANT MEASURES FOR MARKOV PROCESSES

BY

S. HOROWITZ(1)

ABSTRACT

Sufficient conditions are given for the existence of σ-finite invariant measure for conservative and ergodic Markov processes.

1. **Definitions and notations.** A Markov process is defined to be a quadruple (X,Σ,m,P) where (X,Σ,m) is a measure space with finite positive measure m ($m(X)=1$) and where P is an operator on $L_1(m)$ satisfying:

(i) *P is a contraction*: $\|P\| \leqq 1$

(ii) *P is positive*: if $0 \leqq f \in L_1(m)$ then $fP \geqq 0$.

The operator adjoint to P is defined in $L_\infty(m)$. It will also be denoted by P but will be written to the left of its variable. Thus $\langle fP,\, g\rangle = \langle f,\, Pg\rangle$ for $f \in L_1(m)$, $g \in L_\infty(m)$.

The operator P on $L_1(m)$ acts on the signed measure $\lambda \prec m$ as follows:

$$\lambda P(A) = \int P1_A \lambda(dx). \tag{1.1}$$

Equation (1.1) will occasionally be used for σ-finite positive measures.

We shall also define the operator I_A, for $A \in \Sigma$, by

$$I_A f(x) = 1_A(x) f(x) \tag{1.2}$$

$$\lambda I_A(B) = \lambda(B \cap A). \tag{1.3}$$

The process is said to be *ergodic* if

$$0 < m(A) < 1 \Rightarrow P1_A \neq 1_A. \tag{1.4}$$

Let us define the operator:

$$P_A = I_A \sum_{n=0}^{\infty} (PI_{A^c})^n PI_A. \tag{1.5}$$

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It is well known that (A, Σ, m, P_A) is a Markov process.

The process (X, Σ, m, P) is said to be *conservative* if $P_A 1_A = 1_A$ for all A with $m(A) > 0$.

It can be shown (see for example [1]) that if the process is ergodic and conservative then for every non-zero function $0 \leqq f \in L_\infty(m)$.

$$\sum_{n=1}^{\infty} P^n f(x) = \infty \text{ a.e.} \tag{1.6}$$

A positive measure μ is called *invariant* (under P) if (1.7) $\mu P = \mu$.

Throughout this paper, (X, Σ, m, P) is assumed to be an ergodic and conservative Markov process.

2. On existence of a σ-finite invariant measure.

THEOREM 1. *The condition* (2.1) *is sufficient for the existence of a σ-finite invariant measure μ, equivalent to m, such that $\mu(A) = 1$ for some set $A \in \Sigma$.*

$$\text{If } m(B) > 0 \text{ then } \limsup_{N \to \infty} \frac{\sum_{n=1}^{N} mP^n(B)}{\sum_{n=1}^{N} mP^n(A)} > 0. \tag{2.1}$$

THEOREM 2. *Let us denote*

$$\psi_N(x, A, B) = \frac{\sum_{n=1}^{N} P^n 1_B(x)}{\sum_{n=1}^{N} P^n 1_A(x)}. \tag{2.2}$$

The condition (2.3) *is sufficient for the existence of a σ-finite invariant measure μ, equivalent to m such that $\mu(A) = 1$.*

$$\limsup_{M \to \infty} \frac{1}{M} \sum_{N=1}^{M} \psi_N(x, A, B) > 0 \qquad (m(B) > 0) \tag{2.3}$$

for all $x \in E$ where $m(E) > 0$ (E depends on B).

REMARK. If there exists such a measure, then by the Chacon-Ornstein Theorem:

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N} P^n 1_B(x)}{\sum_{n=1}^{N} P^n 1_A(x)} = \mu(B) \text{ a.e.}$$

Hence, the condition of Theorem 2 is also necessary.

LEMMA 1. *Let λ be a finite measure invariant under P_A, then*

$$\mu = \lambda I_A \sum_{n=0}^{\infty} (PI_A)^n$$

is σ-finite measure on x, invariant under P, and $B \subset A \Rightarrow \mu(B) = \lambda(B)$.

Proof. This is Lemma 1 of [3].

LEMMA 2. *For each integer N, and $0 \leqq f \in L_\infty(m)$:*

$$\sum_{n=1}^{N} P^n P_A f(x) \leqq \sum_{n=1}^{N} P^n I_A f(x) + \|f\|_\infty. \tag{2.4}$$

Proof. Let N be fixed, for each integer K and $0 \leqq f \in L_\infty(m)$:

$$\begin{aligned}
0 &\leqq \sum_{n=1}^{N} P^n I_A \sum_{k=0}^{K} (PI_{A^c})^k PI_A f(x) = \sum_{n=1}^{N} P^n (I - I_{A^c}) \sum_{k=0}^{K} (PI_{A^c})^k PI_A f(x) \\
&= \sum_{n=1}^{N} P^n \sum_{k=0}^{K} (PI_{A^c})^k PI_A f(x) - \sum_{n=1}^{N} P^{n-1} \sum_{k=1}^{K+1} (PI_{A^c})^k PI_A f(x) \\
&= \left(\sum_{n=1}^{N} P^n \sum_{k=0}^{K} (PI_{A^c})^k PA_A f(x) - \sum_{n=1}^{N} P^{n-1} \sum_{k=0}^{K} (PI_{A^c})^k PI_A f(x) \right) \\
&\quad + \sum_{n=1}^{N} P^{n-1} \left(\sum_{k=0}^{K} (PI_{A^c})^k - \sum_{k=1}^{K+1} (PI_{A^c})^k \right) PI_A f(x) \\
&= P^N \sum_{k=0}^{k} (PI_{A^c})^k PI_A f(x) - \sum_{k=0}^{K} (PI_{A^c})^k PI_A f(x) \\
&\quad + \sum_{n=1}^{N} P^n I_A f(x) - \sum_{n=1}^{N} P^{n-1} (PI_{A^c})^{K+1} PI_A f(x) \\
&\leqq \sum_{n=1}^{N} P^n I_A f(x) + P^N \sum_{k=0}^{K} (PI_{A^c})^k PI_A f(x) \leqq \sum_{n=1}^{N} P^n I_A f(x) + \|f\|_\infty.
\end{aligned}$$

But this inqualiity is true for every K, let $K \to \infty$ and then we get $\sum_{n=1}^{N} P^n P_A f(x) \leqq \sum_{n=1}^{N} P^n I_A f(x) + \|f\|_\infty$.

Let us define a functional v on $L_\infty(A, \Sigma, mI_A)$ as follows: Let $\{N_j\}$ be a sequence of integers:

$$v(f) = \operatorname*{Lim}_{j} \left[\frac{\sum_{n=1}^{N_j} \langle mP^n, f \rangle}{\sum_{n=1}^{N_j} mP^n(A)} \right] \quad \text{(a Banach limit).} \tag{2.5}$$

Let us also define an operator T from $L_\infty(A, \Sigma, mI_A)$ into $L_\infty(X, \Sigma, m)$ as follows:

$$Tf(x) = \operatorname*{Lim}_N \left[\frac{\sum_{n=1}^{N} P^n f(x)}{\sum_{n=1}^{N} P^n I_A(x)} \right] \tag{2.6}$$

it is clear that $\| v \| = 1$ and $\| T \| = 1$.

LEMMA 3. *For each $f \in L_\infty(A,\Sigma,m)$ we have:*

$$Tf(x) = TP_A f(x) \tag{2.7}$$

$$v(f) = v(P_A f). \tag{2.8}$$

Proof. According to (2.3) and (2.6) and by the fact that $\sum_{n=1}^{\infty} P^n 1_A(x) = \infty$ a.e. we have for each $0 \leqq f \leqq 1_A$: $Tf(x) \geqq TP_A f(x)$ a.e. but $T1_A - TP_A f = TP_A(1_A - f) \leqq T1_A - Tf \Rightarrow Tf \leqq TP_A f$. Hence $Tf(x) = TP_A f(x)$ a.e. and (2.7) is proved. The proof of (2.8) is similar.

LEMMA 4. *If there is no σ-finite invariant measure μ equivalent to m such that $\mu(A) = 1$ then there is a non-zero function $0 \leqq g \leqq 1_A$ and a sequence of intergers $\{n_i\}$ so that*

$$\sum_{i=1}^{\infty} P_A^{n_i} g \leqq 1_A. \tag{2.9}$$

Proof. Lemma 3 of [6] says: Let (X, Σ, m, P) be a Markov process and there is *no* finite measure invariant under P then there is a non-zero function $0 \leqq g \leqq 1$ and a sequence of integers $\{n_i\}$ so that $\sum_{i=1}^{\infty} P^{n_i} g \leqq 1$.

Now, if there is *no* σ-finite invariant measure μ such that $\mu(A) = 1$, then we can conclude from Lemma 1 that there is no finite measure, supported on A and invariant under P_A. Hence, there is a function $0 \leqq g \leqq 1_A$ and a sequence of integers $\{n_i\}$ so that $\sum_{i=1}^{\infty} P_A^{n_i} g \leqq 1_A$.

Proof of Theorem 1. Let us assume that there is *no* σ-finite invariant measure μ equivalent to m such that $\mu(A) = 1$. Let g be the function of Lemma 4.

By (2.1), there can be found a sequence of integers $\{N_j\}$ so that:

$$\lim_{j \to \infty} \frac{\sum_{n=1}^{N_j} \langle mP^n, g \rangle}{\sum_{n=1}^{N_j} mP^n(A)} > 0.$$

Let us put this sequence in (2.5), and then $v(g) > 0$. But, by (2.8), and (2.9) we have, for every integer N:

$$1 = \nu(1_A) \geqq \nu\left(\sum_{i=1}^{N} P_A^{n_i} g\right) = N\nu(g),$$

A contradiction. So, Theorem 1 is proved.

Proof of Theorem 2. Let $B = \{x \mid g(x) > \varepsilon\}$ where g is the function of Lemma 4, $B \subset A$, and we can find a $\varepsilon > 0$ so that $m(B) > 0$, hence $\sum_{i=1}^{\infty} P_A^{n_i} 1_B \leqq 1/\varepsilon 1_A$. By (2.7) and (2.9) we have for every integer N: $(1/\varepsilon)1_A = (1/\varepsilon)T1_A \geqq \sum_{i=1}^{N} TP_A^{n_i} 1_B = N \cdot T1_B$. Hence, $T1_B = 0$, or $\mathrm{Lim}\, \psi_N(x, A \cdot B) = 0$ for all Banach limits, where $\psi_N(x, A, B)$ is defined in (2.2). That means, the sequence $\psi_N(x, A, B)$ almost converges to zero, and by Theorem 1 of [5]: $\frac{1}{M} \sum_{N=1}^{M} \psi_N(x, A, B) \underset{M \to \infty}{\to} 0$ a.e. A contradiction to (2.3). So Theorem 2 is proved.

REMARK. Without the assumption that the process is ergodic, the conditions of Theorem 1 are sufficient to show the existence of a σ-finite invariant measure μ, with $\mu(A) = 1$, supported on $\tilde{A}$ and equivalent to $mI\tilde{A}$, where

$$\tilde{A} = \left\{x \,\middle|\, \sum_{n=1}^{\infty} P^n 1_A(x) > 0\right\}.$$

THEOREM 3. (a) *Let B be a set with $m(B) > 0$ and*

$$\lim_{M \to 0} \frac{1}{M} \sum_{N=1}^{M} \psi_N(x, A, B) = 0 \quad \text{a.e.}$$

then $X = \bigcup_{j=1}^{\infty} B_j$ so that

$$\lim_{M \to 0} \frac{1}{M} \sum_{N=1}^{M} \psi_N(x, A, B_j) = 0 \text{ a.e. } \textit{for all } j. \tag{2.10}$$

(b) *Let B be a set with $m(B) > 0$ and*

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N} mP^n(B)}{\sum_{n=1}^{N} mP^n(A)} = 0$$

then $X = \bigcup_{j=1}^{\infty} B_j$ so that

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N} mP^n(B_j)}{\sum_{n=1}^{N} mP^n(A)} = 0 \textit{ for all } j. \tag{2.11}$$

Proof. Let us define:

$$B_{ki} = \left\{ x \,\middle|\, P^k 1_B(x) \geqq \frac{1}{i} \cdot \right\} \tag{2.12}$$

It is clear that $\bigcup_{k,i} B_{ki} = X$, and $i \cdot P^k 1_B \geqq 1_{B_{ki}}$. Hence:

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} mP^{nn}(B_{ki})}{\sum_{n=1}^{N} mP^n(A)} \leqq \lim_{N\to\infty} \frac{i \cdot \sum_{n=1}^{N} mP^{n+k}(B)}{\sum_{n=1}^{N} mP^n(A)} = 0$$

and (2.11) is proved. The proof of (2.10) is similar: According to the assumption of the theorem, for every $x \in X$, except of a null set, we have: (i) $\sum_{n=1} P^n 1_A(x) = \infty$.

(ii) It can be found a sequence of integers of density 1 $\{N_j\}$ such that $\lim_{j\to\infty} \psi_{Nj}(x,A,B) = 0$, but:

$$\lim_{j\to\infty} \psi_{Nj}(x,A,B_{ik}) \leqq \lim_{j\to\infty} \frac{i \cdot \sum_{n=1}^{Nj} P^{n+k} 1_B(x)}{\sum_{n=1}^{Nj} P^n 1_A(x)} = 0.$$

Hence, $\lim_{M\to\infty} \frac{1}{M} \sum_{N=1}^{M} \psi_N(x,A,B_{ik}) = 0$, and (2.10) is proved.

COROLLARY. *Each one of the following conditions,* (a) *and* (b), *is sufficient for the existence of a σ-finite invariant measure μ such that $\mu(A) = 1$.*

(a) *There can be found a positive number ε, so that for every set B with $m(B) > 1 - \varepsilon$ we have* $\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} mP^n(B)}{\sum_{n=1}^{N} mP^n(A)} > 0.$

(b) *There can be found a positive number ε, so that for every set B with $m(B) > 1 - \varepsilon$ we have* $\limsup_{M\to\infty} \frac{1}{M} \sum_{N=1}^{M} \psi_N(x,A,B) > 0$ *for all $x \in E$, $m(E) > 0$ (E depends on B).*

Proof. By Theorem 2, we can conclude that conditions (a) and (b) of the Corollary imply (2.1) and (2.2) respectively.

THEOREM 4. *Let us assume that the σ-field Σ has an atom A, then there exists a σ-finite invariant measure μ, equivalent to m, such that $\mu(A) = 1$.*

Proof. Let $m(A) = \varepsilon > 0$, then for all $B \in \Sigma$, $m(B) > 1 - \varepsilon \Rightarrow A \subset B$.

Hence: $\dfrac{\sum_{n=1}^{N} mP^n(B)}{\sum_{n=1}^{N} mP^n(A)} \geqq 1$

and by the Corollary of Theorem 2, there is a σ-finite invariant measure μ such that $\mu(A) = 1$.

REMARK. If X is countable then each $x \in X$ is an atom, hence there is a σ-finite invariant measure μ, such that $\mu\{x\} = 1$.

Let $P^n = Q_n + R_n$ where Q_n is an integral operator with the kernel $\omega_n(x, y)$, and if K is any integral operator so that $0 \leqq K \leqq R_n$ then $K = 0$.

DEFINITION. *(X, B, m, P) is said to be a Harris process if $Q_n > 0$ for some integer n.*

The following theorem was proved by Harris [3] (see also [2] and [4]). We shall get it as a consequence of the Corollary of Theorem 2.

THEOREM 5. *If P is a Harris process then there exists a σ-finite invariant measure equivalent to m.*

Proof. Let P be a Harris process where there is an integer k so that $Q_k > 0$, let $\omega_k(x, y)$ be the integral kernel of Q_k. Hence, there are two positive numbers ε, δ, so that if we shall define

$$E_x = \{y \,|\, \omega_k(x, y) > \varepsilon\}$$

then it can be found a set A with $m(A) > 0$, so that $m(E_x) > \delta$ for each $x \in A$.

Let B be a set with $m(B) > 1 - \delta/2$ then $x \in A \Rightarrow m(B \cap E_x) > \delta/2$, and therefore $x \in A \Rightarrow P^k 1_B(x) \;\; \int_{B \cap Ex} \omega_k(x, y) m(dy) \geqq \varepsilon m(B \cap E_x) > (\varepsilon\delta/2)$ and hence $P^k 1_B(x) > (\varepsilon\delta/2) 1_A(x)$, and we get

$$\limsup_{M \to \infty} \frac{1}{M} \sum_{N=1}^{M} \psi_N(x, B, A) \geqq \liminf_{N \to \infty} \frac{\sum_{n=1}^{N} P^{n+k} 1_B(x)}{\sum_{n=1}^{N} P^n 1_A(x)}$$

$$\geqq \liminf_{N \to \infty} \frac{\varepsilon\delta}{2} \cdot \frac{\sum_{n=1}^{N} P^n 1_A(x)}{\sum_{n=1}^{N} P^n 1_A(x)} > \frac{\varepsilon\delta}{2} > 0$$

and by the Corollary of Theorem 3, there is a σ-finite invariant measure, and the theorem is proved.

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The Hebrew University of Jerusalem